Tan Lei and Shishikura’s example of obstructed polynomial mating without a levy cycle.

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Feb. 2012
Origins

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Obstructed mating

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Topological (instant) mating

$K(P_1) \sqcup K(P_2)/\sim$

with $\sim$: relation generated by identifying endpoints of external rays. A dynamics is well defined thereon.

When is the quotient a sphere?
When is the dynamics conjugated to a rational map?
Since PCF (post-critically finite) rational maps are characterized by Thurston’s theorem, it is tempting to try and guess the Th-equivalence class of a potential mating of $P_1$ and $P_2$. 
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Formal mating

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In good cases, it is unobstructed and Th-equivalent to a rational map and to the topological mating.
Degenerate (assisted) mating

However sometimes the formal mating has a Th-obstruction yet the topological mating is conjugated to a rational map. Rees, Shishikura and Tan Lei have devised a way to detect this on the formal mating and to correct the latter by collapsing some post critical points together, yielding a new ramified cover that is unobstructed, and proved that it is Th-equivalent to a rational map conjugated to the topological mating.
The last case is when the obstruction cannot be removed. Then, the topological mating cannot be equivalent to a rational map (even though the quotient still may be a sphere, or not).
Define a Riemann surface $S_R$ by cutting & pasting along equipotential $e^R$, $R > 1$. Glue according to external angle.
Uniformize to $\hat{\mathbb{C}}$. Here: stereographic projected to $S^2$. 
There is a natural holomorphic map (rational of degree $d$ after uniformization)

$$F_{R} : S_{R} \to S_{R^{d}}.$$
Slow mating

$S_{R^{1/2}}$  $S_{R^2}$

$F_R$  $F_{R^2}$

$F_{R^{1/2}}$  $F_{R^4}$
Question: Do the maps $F_R$ converge as $R \to 1$ to a rational map of the same degree?

It is then tempting to define the latter as a mating of $P_1$ and $P_2$. 
In the PCF case, the post-critical set of $P_1$ and $P_2$ map to Riemann surfaces $S_R$, so we get Riemann surfaces with marked points. The sequence of marked $S_{R^{1/d^n}}$ for $n \in \mathbb{N}$ is an orbit under “Thurston’s pull-back map associated to the formal mating”. 
Comparison

Corrected Formal mating

Th-equiv class of the Formal mating

PCF polyn

Topological mating

$J$ connected and locally connected

Slow mating

$J$ connected
The example

It is a mating of two PCF polynomials of degree 3 whose formal mating has a non removable Th-obstruction.
The example
The example
The example

Matrix of the multicurve \( \{\text{orange, green}\} \):

\[
\begin{bmatrix}
1/2 & 1/2 \\
1 & 0
\end{bmatrix}
\]

Spectrum: \( \{1, 1/2\} \).
Remark: Shishikura and Tan Lei have proved that the ray equivalence relation is closed and that classes are trees with a bounded number of equator crossing: thus the topological mating gives a sphere. Also, the topological mating is Th-equivalent to the formal mating (and thus not to a rational map).
Pinching curves
Show movie.
Three normalizations
Three normalizations
Three normalizations
Three normalizations
Interpretation: limit dynamical system.

There is a limit dynamical system on a tree of spheres: the tree of three spheres obtained when the canonical obstruction gets completely pinched.
Interpretation: limit dynamical system.
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The third iterate of the limit maps each sphere to itself, by three semi-conjugated degree 6 rational maps.
Interpretation: limit dynamical system.
Interpretation: tubes and mess.
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